Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Find $\lim_{n\to\infty}\left((n+1)\left(\frac{a^{\frac{1}{n+1}}+b^{\frac{1}{n+1}}}{2}\right)^{n+1}-n\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^{n}\right)$.

Solution.

First we will prove $\lim_{n\to\infty}\left((n+1)\left(\frac{1+t^{\frac{1}{n+1}}}{2}\right)^{n+1}-n\left(\frac{1+t^{\frac{1}{n}}}{2}\right)^{n}\right)$ for $0<t<1$.

Let $a_n:=\left(\frac{1+t^{\frac{1}{n}}}{2}\right)^n$ and $\alpha_n:=\frac{t^{\frac{1}{n}}-1}{2}$ then $a_n=(1+\alpha_n)^n$,$\lim_{n\to\infty}\alpha_n=0$ and

$\lim_{n\to\infty}\left(\frac{1+t^{\frac{1}{n}}}{2}\right)^n=\lim_{n\to\infty}a_n=\lim_{n\to\infty}(1+\alpha_n)^n=\lim_{n\to\infty}e^{n\ln(1+\alpha_n)}$.

Since $\lim_{n\to\infty}\ln(1+\alpha_n)^n=\lim_{n\to\infty}n\alpha_n\cdot\frac{\ln(1+\alpha_n)}{\alpha_n}=\lim_{n\to\infty}n\alpha_n=\frac{1}{2}\lim_{n\to\infty}n\left(t^{\frac{1}{n}}-1\right)=\ln\sqrt{t}$

then $\lim_{n\to\infty}a_n=e^{\ln\sqrt{t}}=\sqrt{t}$ and, therefore, $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1$, $\lim_{n\to\infty}((n+1)a_{n+1}-na_n)=$

$\lim_{n\to\infty}(a_{n+1}+n(a_{n+1}-a_n))=\sqrt{t}+\lim_{n\to\infty}n(a_{n+1}-a_n)=\sqrt{t}+\lim_{n\to\infty}a_nn\left(\frac{a_{n+1}}{a_n}-1\right)=$

$\sqrt{t}+\sqrt{t}\lim_{n\to\infty}n\left(\frac{a_{n+1}}{a_n}-1\right)$.

We have $\lim_{n\to\infty}n\left(\frac{a_{n+1}}{a_n}-1\right)=\lim_{n\to\infty}\left(n\ln\frac{a_{n+1}}{a_n}\cdot\frac{\frac{a_{n+1}}{a_n}-1}{\ln\frac{a_{n+1}}{a_n}}\right)=\lim_{n\to\infty}\left(n\ln\frac{a_{n+1}}{a_n}\right)=$

$\lim_{n\to\infty}(n(n+1)\ln(1+\alpha_{n+1})-n^2\ln(1+\alpha_n))$.

Since $\ln(1+t)=t-\frac{t^2}{2}+o(t^2)$ then $n(n+1)\ln(1+\alpha_{n+1})-n^2\ln(1+\alpha_n)=$

$n(n+1)\left(\alpha_{n+1}-\frac{\alpha_{n+1}^2}{2}+o(\alpha_{n+1}^2)\right)-n^2\left(\alpha_n-\frac{\alpha_n^2}{2}+o(\alpha_n^2)\right)=n^2(\alpha_{n+1}-\alpha_n)+n\alpha_{n+1}-$

$\frac{n\alpha_{n+1}^2}{2}-\frac{n^2}{2}(\alpha_{n+1}^2-\alpha_n^2)+n(n+1)o(\alpha_{n+1}^2)-n^2o(\alpha_n^2)$.

Noting that $\alpha_n=\frac{t^{\frac{1}{n}}-1}{2}$ asymptotically equivalent to $\frac{\ln\sqrt{t}}{n}$ we obtain

$o(\alpha_{n+1}^2)=o(\alpha_n^2)=o\left(\frac{1}{n^2}\right)$ and, therefore, $\lim_{n\to\infty}(n(n+1)o(\alpha_{n+1}^2)-n^2o(\alpha_n^2))=0$.

Also, we have $\lim_{n\to\infty}n\alpha_{n+1}=\lim_{n\to\infty}(n+1)\alpha_{n+1}=\ln\sqrt{t}$, $\lim_{n\to\infty}n^2(\alpha_{n+1}-\alpha_n)=$

$\frac{1}{2}\lim_{n\to\infty}n^2\left(t^{\frac{1}{n+1}}-t^{\frac{1}{n}}\right)=-\frac{1}{2}\lim_{n\to\infty}n^2\left(t^{\frac{1}{(n+1)n}}-1\right)=-\ln\sqrt{t}$,

$\lim_{n\to\infty}n\alpha_{n+1}^2=\lim_{n\to\infty}n\alpha_{n+1}\cdot\lim_{n\to\infty}\alpha_{n+1}=\ln\sqrt{t}\cdot 0=0$,

$\lim_{n\to\infty}(\alpha_{n+1}+\alpha_n)=\lim_{n\to\infty}\left(t^{\frac{1}{n+1}}+t^{\frac{1}{n}}-2\right)=0$, $\lim_{n\to\infty}n^2(\alpha_{n+1}^2-\alpha_n^2)=$

$\lim_{n\to\infty}n^2(\alpha_{n+1}-\alpha_n)\lim_{n\to\infty}(\alpha_{n+1}+\alpha_n)=(-\ln\sqrt{t})\cdot 0=0$.

Hence, $\lim_{n\to\infty}(n(n+1)\ln(1+\alpha_{n+1})-n^2\ln(1+\alpha_n))=-\ln\sqrt{t}+\ln\sqrt{t}=0$ and, therefore,

$\lim_{n\to\infty}((n+1)a_{n+1}-na_n)=\sqrt{t}$.

Coming back to original problem in assumption $a>b$ and denoting $t:=\frac{b}{a}$, we obtain

$$\lim_{n\to\infty}\left((n+1)\left(\frac{a^{\frac{1}{n+1}}+b^{\frac{1}{n+1}}}{2}\right)^{n+1}-n\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^{n}\right)=$$

$$a\lim_{n\to\infty}\left((n+1)\left(\frac{1+t^{\frac{1}{n+1}}}{2}\right)^{n+1}-n\left(\frac{1+t^{\frac{1}{n}}}{2}\right)^{n}\right)=a\sqrt{t}=\sqrt{ab}.$$

In case $a=b$ we have $\lim_{n\to\infty}\left((n+1)\left(\frac{a^{\frac{1}{n+1}}+b^{\frac{1}{n+1}}}{2}\right)^{n+1}-n\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^{n}\right)=a=\sqrt{a^2}$.

Thus, $\lim_{n\to\infty}\left((n+1)\left(\frac{a^{\frac{1}{n+1}}+b^{\frac{1}{n+1}}}{2}\right)^{n+1}-n\left(\frac{a^{\frac{1}{n}}+b^{\frac{1}{n}}}{2}\right)^{n}\right)=\sqrt{ab}$ for any positive a,b.